Topics in complex analysis Lecture 9 (Nov 18, 2024)

<u>Rem</u> $\mathbb{C}$ is not biholomorphically equiv to $B_1(0)$. Indeed, otherwise let $\varphi : \mathbb{C} \longrightarrow B_1(0)$ be such a biholo map & consider a nonconst bdd holo fct f on $B_1(0)$. Then $f \circ \varphi$ is a nonconst bdd holo fct def on $\mathbb{C}$. By Liouville thm, such a fct doesn't exist.

Lemma 6.1. *Let $f : U \to \mathbb{C}$ be holomorphic and injective. Then $f'(z_0) \neq 0$ for all $z_0 \in U$ and the inverse function $f^{-1} : f(U) \to \mathbb{C}$ is also holomorphic.*

<u>Proof</u> We first show $f'(z_0) \neq 0 \ \forall z_0 \in U$.

Assume to the contrary $f'(z_0) = 0 \ \forall z_0 \in U$.

Up to considering $z \longmapsto f(z+z_0) - f(z_0)$ on $\underbrace{U - z_0}_{\text{translated set}}$

we assume $z_0 = 0 = f(z_0) = f'(z_0)$

Since f is inj., it's $\not\equiv 0$, so by Laurent series expansion $\exists$ minimal $k \in \mathbb{N}$ st $f^{(k)}(z_0) \neq 0$. Then

$$\underset{\text{around } z_0 = 0}{f(z)} = \sum_{n=k}^{\infty} a_n z^n = z^k \underbrace{\sum_{n=0}^{\infty} a_{n+k} z^n}_{=: g(z)} \qquad (*)$$

$g(z)$ holo & $g(0) \neq 0$.

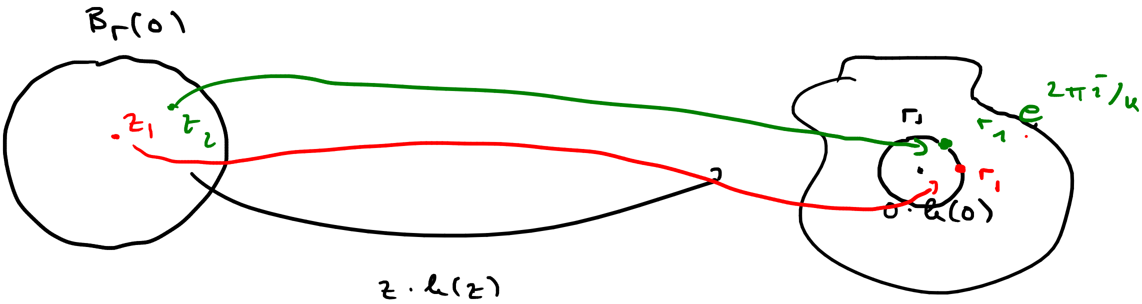
Since g is cts $\exists r > 0$ st $B_r(0) \subseteq U$

& $\inf_{z \in B_r(0)} |g(z)| > 0$.

Now $B_r(0)$ is simply connected & g doesn't assume the value 0 there. By Cor 5.8 $\exists h : B_r(0) \longrightarrow \mathbb{C}$ st $h^k = g$. By $(*)$ this means $f(z) = \left(z h(z)\right)^k$

the fct $z \longmapsto z\, h(z)$ is nonconst (since f is) and holo. Hence by open mapping thm $\exists\, r_1 > 0$

$$\left.\begin{array}{ll} z_1, z_2 \in B_r(0) \text{ st} & z_1\, h(z_1) = r_1 \\ z_1 \neq z_2 & z_2\, h(z_2) = r_1\, e^{2\pi i/k} \end{array}\right\} (*)$$



This is open by open mapping thm

(*) This means $f(z_1) = f(z_2)$ ↯

Let's turn to holo of $f^{-1} : f(U) \longrightarrow \mathbb{C}$.

By open mapping thm $f(U)$ is open. Since $f'(z_0) \neq 0$ the inverse fct thm yields differentiability of the inverse f^{-1} as a map from (a subset of) $\mathbb{R}^2$ to $\mathbb{R}^2$. The Differential of f is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ so its inverse is as well

Cauchy - Riem. ODE.

Hence f^{-1} sats the Cauchy-Riemann ODE, hence f^{-1} is holo $\square$

Theorem 6.3 (Riemann mapping theorem). *Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Then there exists a biholomorphic map $f : G \to B_1(0)$.*

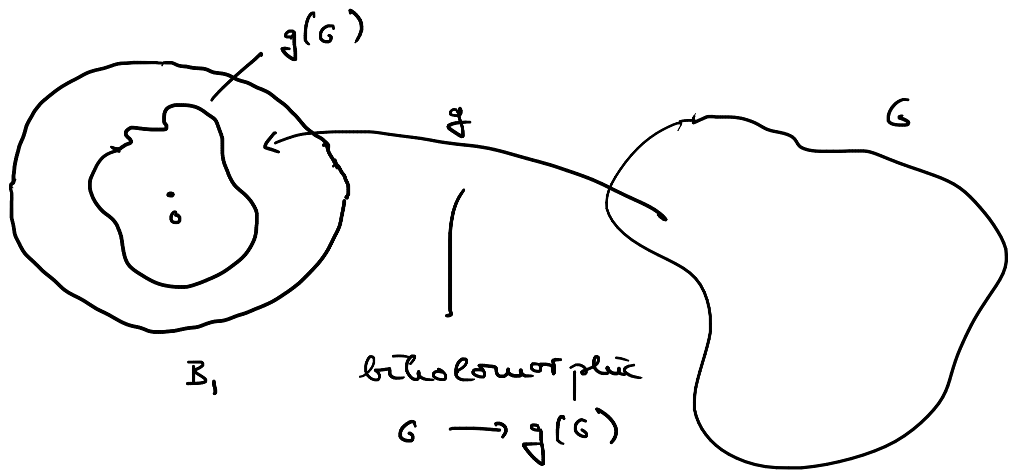
<u>Proof</u> By lemma 6.1 it suffices to find a bijective holomorphic map $f : G \longrightarrow B_1(0)$

(because f^{-1} will be invertible as well)

This is done in several steps.

(1) Reduction of domain. We claim $\exists g$: $G \to B_1(0)$ injective holo and $0 \in g(G)$.

(This will allow us to assume $G \subseteq B_1(0)$ & $0 \in G$)



(2) Ensuring surjectivity. We claim given f: $G \to B_1(0)$ with $G \subseteq B_1(0)$, $0 \in G$, f inj & holo with $f(0) = 0$, then maximality of $|f'(0)|$ implies its surj.

(3) Find a bij. fct by maximizing $|f'(0)|$ among all inj holo fcts $f_0 : G \to B_1(0)$ st. $f(0) = 0$.

<u>Proof of (1)</u> Assume that $\mathbb{C} \setminus G$ contains an open ball $B_{2r}(z_0)$. Then the map

$$g_1 : G \to B_1(0), \quad g_1(z) = r \frac{1}{z - z_0}$$

is well-def., holo, and inj.

Let $z_1 \in g(G) \subseteq \overline{B_{1/2}(0)}$. Then the map

$$g : G \longrightarrow B_1(0), \quad g(z) := \frac{1}{2}\left(g_1(z) - z_1\right)$$

is well-def, holo, inj, and $0 \in g(G)$.

(by considering a preimage of z_1 under g)

In general, $\mathbb{C} \setminus G$ will not contain an open ball.
Hence we have to modify the previous constr.

Since $G \neq \mathbb{C}$, $\exists\, z_0 \in \mathbb{C} \setminus G$. Clearly, $z \longmapsto z - z_0$ doesn't vanish on G. By Cor 5.8, we can consider its $\underset{\text{holo}}{\wedge}$ sqrt $z \longmapsto \sqrt{z - z_0}$

<u>Claim</u> this sqrt is injective.

If $\sqrt{z_1 - z_0} = \sqrt{z_2 - z_0}$, squaring yields $z_1 = z_2$

<u>Claim</u> $\mathbb{C} \setminus \underbrace{\sqrt{G - z_0}}$ contains an open ball.

image of $z \longmapsto \sqrt{z - z_0}$ on G

Indeed, by the open mapping thm $\exists\, \hat{z} \neq 0$ and $r > 0$ st. $B_{2r}(\hat{z}) \subseteq \sqrt{G - z_0}$. We show $-B_{2r}(\hat{z}) \subseteq \mathbb{C} \setminus \sqrt{G - z_0}$. If not $\exists\, z_1, z_2 \in G$ st

$$\sqrt{z_1 - z_0} = -\sqrt{z_2 - z_0} \qquad \left(\text{since } -B_{2r}(\hat{z}) \cap \sqrt{G - z_0} \neq \emptyset\right)$$

by assumption

squaring yields $z_1 = z_2$ but this is possible only for $z_1 = z_0$

But then $z_0 \in G$ $\lightning$

(2) Claim that if $0 \in G \subseteq B_1(0)$ & $f : G \to B_1(0)$ inj holo, $f(0) = 0$ but not surj. then $\exists\, \tilde{f} : G \to B_1(0)$ inj holo $\tilde{f}(0) = 0$ with $|\tilde{f}'(0)| > |f'(0)|$.

Recall given any $z_0 \in B_1(0)$ the map $\varphi_{z_0} : B_1(0) \to \mathbb{C}$

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

is a biholomorphic map onto $B_1(0)$ (surj).

By assumption on f, $\exists\, z_0 \in B_1(0) \setminus f(G)$.

This means $\varphi_{z_0} \circ f(z) \neq 0$ $\forall z \in G$. since φ_{z_0} vanishes only at $z_0 \notin f(G)$.

Again by Cor 5.8 $\exists$ holo sqrt $\sqrt{\varphi_{z_0} \circ f}$, which is inj. and $\sqrt{\varphi_{z_0}(f(0))} = \sqrt{-z_0} \in B_1(0)$. Now we define

$$\tilde{f} := \varphi_{z_1} \circ \sqrt{\varphi_{z_0} \circ f} : G \to B_1(0).$$

where $z_1 = \sqrt{\varphi_{z_0}(f(0))}$.

Check: $\tilde{f}$ inj holo & $\tilde{f}(0) = 0$.

Moreover note $h : B_1(0) \to B_1(0)$ def by

$h := \varphi_{z_0}^{-1} \circ (\varphi_{z_1}^{-1})^2$ holo, not inj., & $h \circ \tilde{f} = f$.

$\Downarrow$

Schwarz lemma: $|h'(0)| < 1$

Chain rule implies

$$|f'(0)| = |h'(\tilde{f}(0))\, \tilde{f}'(0)|$$
$$= |h'(0)|\, |\tilde{f}'(0)| < |\tilde{f}'(0)|.$$

Proof of (3) Find a maximizer of

$$\sup\{ |f'(0)| : f : G \to B_1(0) \text{ holo inj } \\ \& \text{ sats } f(0) = 0 \}.$$

By (2) a maximizer will automatically be surj.
hence will be the desired biholomorphy by Lemma 6.1.

As seen in the exercises; a maximizer exists.
once the sets of admissible fcts is nonempty.

But the latter is the case: since $G \subseteq B_1(0)$
& $0 \in G$, $f(z) = z$ is such a map. $\square$